IF THEY ARE LIMIT PERIODIC?

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Abstract. We prove a partial result concerning the long–standing problem on limit periodicity of the Jacobi matrix associated with the balanced measure on the Julia set of an expending polynomial. Besides this, connections of the problem with the Faybusovich–Gekhtman flow and many other objects (the Hilbert transform, the Schwarz derivative, the Ruelle and Laplace operators) that, we sure, are of independent interest, are discussed.

1. Introduction

In 80's the following interesting phenomena was discovered: the spectral measure of an almost periodic Jacobi matrix can be singular continuous (supported on a Cantor type set of the zero Lebesgue measure). The effect was studied from both sides — from coefficient sequences to spectral data [1], [4] and from spectral data to Jacobi matrices.

The second, usually more elegant, approach produced the following example [3], [2]. Let $T(z) = z^2 - C$. For $C > 2$ the Julia set E of T is a real Cantor type set, $|E| = 0$. Denote by μ the balanced measure on E, $\mu(T^{-1}(F)) = \mu(F)$ for all $F \subset E$. Let

$$J = \begin{bmatrix} q_0 & p_1 & & \\ p_1 & q_1 & p_2 & \\ & \ddots & \ddots & \ddots \end{bmatrix} \tag{1}$$

be the Jacobi matrix associated to the given measure. Note that to construct $J : l^2(\mathbb{Z}_+) \to l^2(\mathbb{Z}_+)$ one uses the three term recurrent relation for polynomials orthonormal in $L^2_{d\mu}$

$$\lambda P_k(\lambda) = p_k P_{k-1}(\lambda) + q_k P_k(\lambda) + p_{k+1} P_{k+1}(\lambda), \tag{2}$$

of course $P_k \mapsto |k\rangle$, where $\{|k\rangle\}$ is the standard basis in $l^2(\mathbb{Z}_+)$.

Then the given matrix satisfies the renormalization equation:

$$V^* T(J) V = J,$$

where $V|k\rangle = |2k\rangle$. In fact, this is a system of nonlinear equations for p_n's ($q_n = 0$ in this case), due to which at least for $C > 3$ one gets inductively that

$$|p_{2^n l+m} - p_m| \le \epsilon_n, \quad \text{for all } \ l, m; \quad \epsilon_n \to 0 \ (n \to \infty).$$

That is the sequence $\{p_n\}$ and, by definition the matrix itself, is limit periodic. It looks very natural to conjecture that if only T is an arbitrary expanding polynomial in the sense of Complex Dynamics [7] then its balanced measure produces a limit periodic Jacobi matrix. Several research groups attacked this problem (in full

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generality) but failed. Even the case of the quadratic polynomial with $C > 2$ is still open.

Recall some properties of Jacobi matrices. Let J be a Jacobi matrix, $J^* = J$, acting in $\mathbb{C}^d$ or $l^2(\mathbb{Z}_+)$. Under the assumption $p_k \neq 0$ the vector $|0\rangle$ of the standard basis is cyclic for J. The resolvent function is a function of the form

$$r(z) = \left\langle 0 \left| (J - z)^{-1} \right| 0 \right\rangle . \tag{3}$$

It has positive imaginary part in the upper half plane and hence possesses the representation

$$r(z) = \int \frac{d\sigma}{\lambda - z} = \left\langle \mathbf{1} \left| (\lambda - z)^{-1} \right| \mathbf{1} \right\rangle_{L^2_{d\sigma}} . \tag{4}$$

where $\lambda \cdot$ is the operator multiplication by the independent variable in $L^2_{d\sigma}$ and $\mathbf{1}$ is the function that equals one identically. Formulas (3) and (4) give one to one correspondence between triples $\{L^2_{d\sigma}, \lambda\cdot, \mathbf{1}\}$ and $\{l^2(\mathbb{Z}_+), J, |0\rangle\}$ or $\{\mathbb{C}^d, J, |0\rangle\}$, respectively, in the finite dimensional case. To restore J starting from the nonnegative measure σ one uses (2).

Our first object is the following

Conjecture 1.1. *Let $T(z)$ be an expanding polynomial of degree d with a real Julia set E, $E \subset [-\xi, \xi]$, $T^{-1} : [-\xi, \xi] \to [-\xi, \xi]$. Define $J = J(x)$ by*

$$\left\langle 0 \left| (z - J(x))^{-1} \right| 0 \right\rangle = \frac{T'(z)/d}{T(z) - x}, \quad x \in [-\xi, \xi]. \tag{5}$$

Respectively $J_n(x)$ is associated with an iteration $T_n = T^{\circ n}$, $\deg T_n = d_n$. Then for every ϵ there exists n such that

$$||J_n(x) - J_n(0)|| \leq \epsilon. \tag{6}$$

Note that eigenvalues of $J_n(x)$ and $J_n(0)$ are close, so the non trivial part deals with eigenvectors.

Let us explain how this conjecture is related to the general one. If μ is the balanced measure on E, then the resolvent of $J = J(\mu)$ satisfies to the following Renormalization Equation

$$V^*(z - J)^{-1} V = (T(z) - J)^{-1} T'(z)/d, \tag{7}$$

where $V|k\rangle = |kd\rangle$. Let us include J into a chain $\{J_n(t)\}_{t \in [0,1]}$ defined by

$$V_n^*(z - J_n(t))^{-1} V_n = (T_n(z) - tJ)^{-1} T_n'(z)/d_n$$

(compare the last equation with (5)). Then the main goal is to show that

$$||J_n(1) - J_n(0)|| \leq \epsilon \quad \text{for } n > n_0,$$

since it would imply immediately that $J(\mu)$ is limit periodic. Thus to prove Conjecture 1.1 is a good model problem on the way to prove limit periodicity of $J(\mu)$.

The following approach looks very natural: to get (6) we have to estimate $J'(x)$. The given derivative has a special representation

$$\frac{dJ(x)}{dx} = F(J) + [G, J] \tag{8}$$

with $F(J) = \{T'(J)\}^{-1}$. It is a certain flow on Jacobi matrices that in a sense is dual to the well–known Toda flow. We call it FG flow [6] (see Sect. 2). The first term at the right hand side in (8) is small due to the characteristic property of

expanding polynomials: $|T_n'(x)| \geq Ac^n$, $x \in E$, with $A > 0$, $c > 1$. It appears that the estimation we get for G is not enough to state that the commutator $[G, J]$ is sufficiently small (Proposition 2.6). However on this way we found quite designing formulas and connections with so many objects (the Hilbert transform, the Schwarz derivative, the Ruelle and Laplace operators) that, we sure, they are of independent interest.

In the framework of this approach, initiated in [9], we managed to prove the following theorem that partially confirms the main hypothesis.

Theorem 1.2. *Let J be the Jacobi matrix associated with iterations of an expanding polynomial T. Then for every ϵ there exists n such that*

$$|p_{k+sd_n^2} - p_k| \leq \epsilon, \quad |q_{k+sd_n^2} - q_k| \leq \epsilon, \tag{9}$$

for all $s \geq 0$ and $k = 1, 2, ..., d_n$.

Note that actually our goal is to prove (9) when $k = 1, 2, ..., d_n^2$. A proof of the theorem is given in Sect. 3.

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2. FG flow

2.1. **Definition.** Let $J : \mathbb{C}^d \to \mathbb{C}^d$. Consider the resolvent function

$$\left\langle 0 \left| (z - J)^{-1} \right| 0 \right\rangle = \sum_{k=1}^{d} \frac{\sigma_k}{z - \lambda_k}. \tag{10}$$

Under the Toda flow the spectrum is stable $\lambda_k = \text{Const}$ but masses vary with time $\sigma_k = \sigma_k(t)$. In FG flow case $\lambda_k = \lambda_k(t)$ but $\sigma_k = \text{Const}$. Moreover, in our case (5) time is x, $\sigma_k = 1/d$ and $T(\lambda_k(x)) = x$. Recall $T(z)$ is an expanding polynomial of degree d with a real Julia set E, $E \subset [-\xi, \xi]$, $T^{-1} : [-\xi, \xi] \to [-\xi, \xi]$.

We want to get a differential equation on J. Let $\mathfrak{B}$ be a unitary matrix such that

$$J\mathfrak{B} = \mathfrak{B}\Lambda,$$

where $\Lambda = \text{diag}\{\lambda_k\}$. Since we can choose

$$\lambda_1(x) < \lambda_2(x) < ... < \lambda_d(x)$$

that holds for all x, $\mathfrak{B}$ essentially is well defined. We put

$$\mathfrak{B} = \frac{1}{\sqrt{d}} \begin{bmatrix} P_0(\lambda_1) & ... & P_0(\lambda_d) \\ \vdots & & \vdots \\ P_{d-1}(\lambda_1) & ... & P_{d-1}(\lambda_d) \end{bmatrix},$$

where $P_k(z)$ is the orthonormal polynomial.

We differentiate J with respect to x

$$\dot{J} = \mathfrak{B}\dot{\Lambda}\mathfrak{B}^{-1} + \dot{\mathfrak{B}}\Lambda\mathfrak{B}^{-1} - \mathfrak{B}\Lambda\mathfrak{B}^{-1}\dot{\mathfrak{B}}\mathfrak{B}^{-1} = F + GJ - JG,$$

where $F := \mathfrak{B}\dot{\Lambda}\mathfrak{B}^{-1}$, $G := \dot{\mathfrak{B}}\mathfrak{B}^{-1}$. By the definition $F = f(J)$ with $f(\lambda_k) = \dot{\lambda}_k$. Thus $F = T'(J)^{-1}$. The next step is to determine G.

Note some evident facts. G is skew–symmetric and $\langle 0|G = 0$, so $G|0\rangle = 0$. Also it is easy to show, say by induction, that

$$\frac{d}{dx} J^n = nJ^{n-1}F + GJ^n - J^n G.$$

Finally, since $P_k(J)|0\rangle = |k\rangle$ and

$$\frac{d}{dx} P_k(J) - \frac{\partial}{\partial x} P_k(J) = FP_k'(J) + GP_k(J) - P_k(J)G$$

we get

$$-\frac{\partial}{\partial x} P_k(J)|0\rangle = FP_k'(J)|0\rangle + GP_k(J)|0\rangle. \tag{11}$$

Let G_+ be a lower triangle matrix with zeros on the main diagonal such that $G = G_+ - G_+^*$. Then (11) implies

$$G_+|k\rangle = G_+P_k(J)|0\rangle = -(FP_k'(J)|0\rangle)_+^{(k)}. \tag{12}$$

Here $h_+^{(k)}$ means that in a vector $h = \{h_j\}_{j=0}^{d-1}$ we have to replace all coordinates h_j, $0 \le j \le k$, by zeros.

Let us rewrite (12) in other words. Define an operator D by

$$D|k\rangle = DP_k(J)|0\rangle := P_k'(J)|0\rangle.$$

Then

$$G_+ = -(FD)_+.$$

It is easy to check using the functional representation in $L^2_{d\sigma}$ that

$$DJ - JD = I - |(p_dP_d)'\rangle\langle P_{d-1}|, \tag{13}$$

where $p_dP_d(\lambda)$ is defined by (2). Note that $p_dP_d(z) = 0$ in $L^2_{d\sigma}$, that is it has the same roots $\{\lambda_k(x)\}$ as $T(z) - x$. Thus $p_dP_d(z) = C(T(z) - x)$ and $(p_dP_d)'(z) = CT'(z)$.

Definition 2.1. *FG flow is given by a differential equation of the form*

$$\dot{J} = F + GJ - JG \tag{14}$$

with $F = f(J)$ and $G = G_+ - G_+^*$, where $G_+ = -(FD)_+$ and D is an (upper triangle) matrix such that commutant $[D, J]$ equals the unity matrix up to a one dimensional perturbation [6].

2.2. (FD) **as a Hilbert transform.**

Lemma 2.2. *The matrix of the operator (FD) with respect to the basis of eigenvectors of J has the form*

$$\begin{bmatrix} \frac{1}{2}\frac{T''(\lambda_1)}{T'(\lambda_1)} & \cdots & \frac{1}{\lambda_1-\lambda_d} \\ \vdots & & \vdots \\ \frac{1}{\lambda_d-\lambda_1} & \cdots & \frac{1}{2}\frac{T''(\lambda_d)}{T'(\lambda_d)} \end{bmatrix} \begin{bmatrix} \frac{1}{T'(\lambda_1)} & & \\ & \ddots & \\ & & \frac{1}{T'(\lambda_d)} \end{bmatrix}. \tag{15}$$

Proof. Let us evaluate D in the basis of eigenvectors of J. In this basis

$$|P(x)\rangle \to \frac{1}{\sqrt{d}} \begin{bmatrix} P(\lambda_1) \\ \vdots \\ P(\lambda_d) \end{bmatrix}.$$

As we know $(p_d P_d)'(\lambda_k) = CT'(\lambda_k)$. Taking into account that now J is diagonal we conclude that the diagonal entries of $DJ - JD$ are zeros. Therefore $P_{d-1}(\lambda_j) = d/\{CT'(\lambda_j)\}$. Thus the right hand side of (13) is of the form

$$I - \begin{bmatrix} T'(\lambda_1) \\ \vdots \\ T'(\lambda_d) \end{bmatrix} \begin{bmatrix} \frac{1}{T'(\lambda_1)}, & \dots, & \frac{1}{T'(\lambda_d)} \end{bmatrix}.$$

Referring again to a diagonal form of J we solve (13) and get

$$D_{ij} = \frac{1}{\lambda_i - \lambda_j} \frac{T'(\lambda_i)}{T'(\lambda_j)}, \quad i \neq j. \tag{16}$$

To find diagonal entries D_{ii} we have to use $D|0\rangle = 0$. Since

$$|0\rangle \to \mathbf{1} \to \frac{1}{\sqrt{d}} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

we get

$$D_{ii} = T'(\lambda_i) \sum_{k \neq i} \frac{1}{T'(\lambda_k)} \frac{1}{\lambda_k - \lambda_i}.$$

Note that

$$\sum_{k=1}^{d} \frac{1}{T'(\lambda_k)} \frac{1}{\lambda_k - z} = -\frac{1}{T(z) - x}.$$

Therefore

$$\begin{aligned} \frac{D_{ii}}{T'(\lambda_i)} &= \lim_{z \to \lambda_i} \left\{ -\frac{1}{T'(\lambda_i)} \frac{1}{\lambda_i - z} - \frac{1}{T(z) - x} \right\} \\ &= \lim_{z \to \lambda_i} \frac{\frac{T(z)-x}{z-\lambda_i} \frac{1}{T'(\lambda_i)} - 1}{T(z) - x} = \frac{\frac{1}{2} T''(\lambda_i)}{(T'(\lambda_i))^2}. \end{aligned} \tag{17}$$

Thus (16) and (17) finish the proof. □

2.3. **Trace of $(FD)^*(FD)$.**

Lemma 2.3. *Let L_2 be a Ruelle operator of the form*

$$L_2 g(x) = \frac{1}{d} \sum_{Ty = x} \left(\frac{g}{T'^2} \right)(y) \tag{18}$$

and let $S(T)$ be the Schwarz derivative of T, $S(T) = \frac{T'''}{T'} - \frac{3}{2} \left(\frac{T''}{T'} \right)^2$. Then

$$\frac{1}{d} \mathrm{tr}\{(\mathrm{FD})^*(\mathrm{FD})\} = -\frac{1}{3} \mathrm{L}_2\{\mathrm{S(T)}\}.$$

Proof. First we simplify

$$u_i = \sum_{k \neq i} \frac{1}{(\lambda_i - \lambda_k)^2} = \lim_{z \to \lambda_i} \left\{ \sum_{T(\lambda) = x} \frac{1}{(z - \lambda)^2} - \frac{1}{(z - \lambda_i)^2} \right\}. \tag{19}$$

Note that

$$\sum_{T(\lambda) = x} \frac{1}{(z - \lambda)} = \frac{T'(z)}{T(z) - x}.$$

That is

$$\sum_{T(\lambda)=x} \frac{1}{(z-\lambda)^2} = \frac{T'^2(z) - T''(z)(T(z)-x)}{(T(z)-x)^2}.$$

So passing in a usual way to the limit in (19) we get

$$u_i = \frac{\frac{1}{2}(T'')^2 - \frac{2}{3}T'T'''}{2(T')^2}(\lambda_i). \tag{20}$$

This means that a diagonal entry of the operator $(FD)^*(FD)$ with respect to the basis of eigenvectors of J has the form

$$\frac{1}{T'^2}\left\{\left(\frac{1}{2}\frac{T''}{T'}\right)^2(\lambda_i) + u_i\right\} = \frac{1}{T'^2}\frac{(T'')^2 - \frac{2}{3}T'T'''}{2(T')^2}(\lambda_i) = -\frac{1}{3}\left(\frac{S(T)}{T'^2}\right)(\lambda_i).$$

□

Naturally, in the same way we can find off diagonal entries of the matrix of the operator $(FD)^*(FD)$.

Lemma 2.4. *For $i \neq j$*

$$\{(FD)^*(FD)\}_{ij} = \frac{1}{T'(\lambda_i)}\frac{2}{(\lambda_i-\lambda_j)^2}\frac{1}{T'(\lambda_i)}.$$

We would consider $\Delta := (FDF^{-1})^*(FDF^{-1})$ as a counterpart of Laplacian due to the following proposition.

Corollary 2.5. *Δ is a positive operator that satisfies*

$$[J,[J,\Delta]] = 2d|0\rangle\langle 0| - 2.$$

Proof. See Lemma 2.4. □

Our plan to estimate $[G,J]$ in (8) was based on the conjecture $||(FD)_n|| \sim \kappa^n$ with $\kappa < 1$ (typically everything that goes to zero in the subject goes to zero as a geometric progression). Since $(G_n)_+ = -(FD)_{n+}$ that would give an estimation on G:

$$||(G_n)_+|| \sim \kappa^n n \log d,$$

and we are done. However the following proposition shows that $||(FD)_n|| \not\to 0$.

Proposition 2.6. *There exists the limit*

$$\lim_{n\to\infty} \frac{1}{d^n}\mathrm{tr}\{(\mathrm{FD})_\mathrm{n}^*(\mathrm{FD})_\mathrm{n}\} = -\frac{1}{3}(\mathrm{I}-\mathrm{L}_2)^{-1}\mathrm{L}_2\mathrm{S}(\mathrm{T}). \tag{21}$$

Proof. Let us use the Chain Rule for the Schwarz derivative

$$S(T_{n+1}) = S(T_n)\circ TT'^2 + S(T).$$

Since $L_2\{g\circ TT'^2\} = g$ holds for every function g, we have $L_2^{n+1}\{S(T_n)\circ TT'^2\} = L_2^n S(T_n)$ and therefore

$$L_2^{n+1}S(T_{n+1}) = L_2^n S(T_n) + L_2^{n+1}S(T) = L_2 S(T) + ... + L_2^{n+1}S(T). \tag{22}$$

The spectral radius of L_2 less than $1/d^2$ (see Lemma 3.2). So (22) completes the proof. □

Remark. We still believe in the limit periodic property of $J(\mu)$. Recall that we have to estimate not $(FD)_n$ itself but the commutator $[G_n, J_n]$. Probably it worth to mention that the right hand side of the commutant identity for $(FD)_n$,

$$(FD)_n J_n - J_n(FD)_n = F_n - d^n|0\rangle\langle 0|F_n$$

goes to zero in norm (it's again Lemma 3.2). That is asymptotically $(FD)_n$ and J_n commute.

□

3. Partial result in the right direction

3.1. **Renormalization equation.** Let

$$Lg(x) = \frac{1}{d}\sum_{Ty=x} g(y)$$

be a Ruelle operator associated with an expanding polynomial $T(z)$. If $\tilde{J}$ is the Jacobi matrix associated with a measure $\tilde{\sigma}$ supported on E, $\tilde{J} := \tilde{J}(\tilde{\sigma})$, then the Renormalization Equation

$$V^*(z-J)^{-1}V = (T(z)-\tilde{J})^{-1}T'(z)/d, \quad V|k\rangle = |kd\rangle, \tag{23}$$

has a unique solution $J := J(\sigma)$, where $\sigma := L^*(\tilde{\sigma})$ [2], [8]. It follows basically from the identity

$$\left(L\frac{1}{z-y}(g\circ T)(y)\right)(x) = \frac{T'(z)/d}{T(z)-x}g(x)$$

and the functional representations of both operators in $L^2_{d\sigma}$ and $L^2_{d\tilde{\sigma}}$ respectively. Note that (23) becomes (7) if $\tilde{\sigma} = \mu$, since for the balanced measure we have $\mu = L^*(\mu)$.

Lemma 3.1. *Let $J^{(s)}$ be the s-th $d\times d$ block of the matrix J, that is*

$$J^{(s)} = \begin{bmatrix} q_{sd} & p_{sd+1} & & & \\ p_{sd+1} & q_{sd+1} & p_{sd+2} & & \\ & \ddots & \ddots & \ddots & \\ & & p_{sd+d-2} & q_{sd+d-2} & p_{sd+d-1} \\ & & & p_{sd+d-1} & q_{sd+d-1} \end{bmatrix}. \tag{24}$$

Then its resolvent function is of the form

$$\left\langle 0 \left| (z-J^{(s)})^{-1} \right| 0 \right\rangle = \frac{T'(z)/d}{T^{(s)}(z)}. \tag{25}$$

Moreover at the critical points $\{c : T'(c) = 0\}$ the following decomposition in a continued fraction holds true

$$T^{(s)}(c) = T(c) - \tilde{q}_s - \frac{\tilde{p}_s^2}{T(c)-\tilde{q}_{s-1} - \dots}. \tag{26}$$

Proof. We write J as a $d\times d$ block matrix (each block is of infinite size):

$$J = \begin{bmatrix} \mathcal{Q}_0 & \mathcal{P}_1 & & & S_+\mathcal{P}_d \\ \mathcal{P}_1 & \mathcal{Q}_1 & \mathcal{P}_2 & & \\ & \ddots & \ddots & \ddots & \\ & & \mathcal{P}_{d-2} & \mathcal{Q}_{d-2} & \mathcal{P}_{d-1} \\ \mathcal{P}_d S_+^* & & & \mathcal{P}_{d-1} & \mathcal{Q}_{d-1} \end{bmatrix}. \tag{27}$$

Here $\mathcal{P}_k$ (respectively $\mathcal{Q}_k$) is a diagonal matrix $\mathcal{P}_k = \text{diaq}\{p_{k+sd}\}_{s\geq 0}$ and S_+ is the one–sided shift. In this case V^* is the projection on the first block–component.

Using this representation and being well known identity for block matrices

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & * \\ * & * \end{bmatrix},$$

we get

$$\frac{T(z) - \tilde{J}}{T'(z)/d} = z - \mathcal{Q}_0 - \begin{bmatrix} \mathcal{P}_1, & ..., & S_+\mathcal{P}_d \end{bmatrix} \{z - J_1\}^{-1} \begin{bmatrix} \mathcal{P}_1 \\ \vdots \\ \mathcal{P}_d S_+^* \end{bmatrix}, \tag{28}$$

where J_1 is the matrix that we obtain from J by deleting the first block–row and the first block–column in (27). Note that in $(z - J_1)$ each block is a diagonal matrix that's why we can easily get an inverse matrix in terms of orthogonal polynomials.

Let us introduce the following notations: everything related to $J^{(s)}$ has superscript s. For instance: $p_k^{(s)} = p_{sd+k}$, $1 \leq k \leq d$, respectively $P_d^{(s)}$ and $Q_d^{(s)}$ mean orthonormal polynomials of the first and second kind. In this terms equation (28) is equivalent to the two series of scalar relations corresponding to the diagonal and off diagonal entries

$$\frac{T(z) - \tilde{q}_{s+1}}{T'(z)/d} = \frac{P_d^{(s+1)}(z)}{Q_d^{(s+1)}(z)} - p_{ds}^2 \frac{Q_{d-1}^{(s)}(z)/p_{ds}}{Q_d^{(s)}(z)} \tag{29}$$

and

$$\frac{\tilde{p}_{s+1}}{T'(z)/d} = \frac{p_1^{(s)} ... p_d^{(s)}}{z^{d-1} + ...} = \frac{1}{Q_d^{(s)}(z)}. \tag{30}$$

We have to remind (see (25) and (30)) that

$$\frac{Q_d^{(s)}(z)}{P_d^{(s)}(z)} = \frac{z^{d-1} + ...}{z^d + ...} = \frac{T'(z)/d}{T^{(s)}(z)}.$$

Now, due to the Wronskian identity, if $T'(c) = 0$ then

$$-p_{ds} Q_{d-1}^{(s)}(c) = \frac{1}{P_d^{(s)}(c)}. \tag{31}$$

So, combining (29), (30) and (31) we get the recurrence relation

$$T(c) - \tilde{q}_{s+1} = T^{(s+1)}(c) + \frac{\tilde{p}_{s+1}^2}{T^{(s)}(c)} \tag{32}$$

with initial data

$$T^{(0)}(c) = T(c) - \tilde{q}_0.$$

$\square$

3.2. p_{sd_n} are exponentially small.

Lemma 3.2. *Let J be the Jacobi matrix associated with iterations $\{T_n\}_{n\geq 1}$ of an expanding polynomial T, that is $J = J(\mu)$ where $L^*\mu = \mu$. Then*

$$C_-(\rho d)^n p_s \leq p_{sd_n} \leq C_+(\rho d)^n p_s \tag{33}$$

with $C_\pm > 0$ and $0 < \rho < 1/d$.

Proof. We recall that $P_{sd} = P_s \circ T$ and $Q_{sd} = (T'/d)Q_s \circ T$ [2], [8]. We use an interpolation formula

$$\int R\,d\mu = \sum_{y:P_{sd}(y)=0} R(y)\frac{Q_{sd}}{P'_{sd}}(y), \quad \deg R < sd, \tag{34}$$

and the Wronskian identity

$$p_{sd}\{P_{sd-1}Q_{sd} - Q_{sd-1}P_{sd}\} = 1. \tag{35}$$

Substituting (35) in (34) we obtain

$$p_{sd}^2 = \int \{p_{sd}P_{sd-1}\}^2\,d\mu = \sum_{y:P_{sd}(y)=0} \{p_{sd}P_{sd-1}(y)\}^2\frac{Q_{sd}}{P'_{sd}}(y) = \sum_{y:P_{sd}(y)=0} \frac{1}{(Q_{sd}P'_{sd})(y)}.$$

Therefore,

$$\begin{aligned} p_{sd}^2 &= \sum_{x:P_s(x)=0}\sum_{y:T(y)=x} \frac{1}{((T'^2/d)(Q_sP'_s)\circ T)(y)} \\ &= \sum_{x:P_s(x)=0} \frac{1}{(Q_sP'_s)(x)}\left\{\frac{1}{d}\sum_{y:T(y)=x}\frac{d^2}{T'^2(y)}\right\} \\ &= \sum_{x:P_s(x)=0} \{p_sP_{s-1}(x)\}^2\frac{Q_s}{P'_s}(x)\left\{\frac{1}{d}\sum_{y:T(y)=x}\frac{d^2}{T'^2(y)}\right\}. \end{aligned}$$

Now we use the Ruelle version of the Perron–Frobenius theorem [7], [5] with respect to L_2 (18). According to this theorem

$$\frac{1}{\rho^{2n}}L_2^n g \to h(x)\int g\,d\nu,$$

uniformly on x with a certain continuous function $h > 0$ and positive measure ν; ρ^2 is the spectral radius of L_2. Combining this with the interpolation formula we get both–sided estimate (33).

We only have to show that $(\rho d) < 1$. Let $b(z)$ be the complex Green's function of the domain $\overline{\mathbb{C}} \setminus E$ with respect to infinity. Consider the sequence of functions $\{f_n\}_{n\geq 1}$, where $f_n(z) := (b^{d_n-1}P_{d_n-1})(z)$. It is a multiple–valued function in the domain $\overline{\mathbb{C}} \setminus E$ with a single–valued modulus which has a harmonic majorant $u_n(z)$: $|f_n(z)|^2 \leq u_n(z)$. Moreover, $u_n(\infty) = ||P_{d_n-1}||^2_{L^2_{d\mu}} = 1$. We claim that f_n should go to zero pointwise. If not then we can find a subsequence $\{f_{n_k}\}$ that converges to a non trivial function f. However, in this case, $(bf)(z)$ is a non trivial single valued in $\overline{\mathbb{C}} \setminus E$ function, $|(bf)(z)|^2$ has a harmonic majorant and $(bf)(\infty) = 0$. This contradicts to the well-known fact that *analytic* capacity (that is the Lebesgue measure in this case) of E is zero.

Therefore the sequence converges to zero. In particular

$$(b^{d_n-1}P_{d_n-1})(\infty) = \frac{1}{p_1 \dots p_{d_n-1}} = \frac{p_{d_n}}{p_1} \to 0, \quad n \to \infty.$$

But $\frac{p_{d_n}}{p_1} \sim (\rho d)^n$, thus $(\rho d) < 1$. □

Remark 3.3. *Let us mention here that $q_{sd} = q_0$ since*

$$q_{sd} = \int yP_{sd}^2\,d\mu = \int yP_s^2 \circ T\,dL^*\mu = \int (Ly)P_s^2\,d\mu$$

and $Ly = q_0$.

3.3. **The result.** First we prove (undoubtedly well–known and simple)

Lemma 3.4. *Assume that two measures* σ *and* $\tilde\sigma$ *are mutually absolutely continuous. Moreover,* $d\tilde\sigma = f\,d\sigma$ *and* $1-\epsilon \le f \le (1-\epsilon)^{-1}$*. Let us associate with these measures Jacobi matrices* $J = J(\sigma)$*,* $\tilde J = J(\tilde\sigma)$*. Then for their coefficients we have*

$$|\tilde p_s - p_s| \le \frac{\epsilon}{1-\epsilon}||J||.$$

Proof. Let us use an extreme property of orthogonal polynomials,

$$\begin{aligned}\tilde p_1^2 ... \tilde p_s^2 = \int \tilde p_1^2 ... \tilde p_s^2 \tilde P_s^2\, d\tilde\sigma \ge (1-\epsilon)\int \{z^s + ...\}^2\, d\sigma \\ \ge (1-\epsilon) \inf_{\{P = z^s + ...\}} \int P^2\, d\sigma = (1-\epsilon) p_1^2 ... p_s^2.\end{aligned}$$

Similarly

$$p_1^2 ... p_{s-1}^2 \ge (1-\epsilon)\tilde p_1^2 ... \tilde p_{s-1}^2.$$

Therefore

$$\frac{1}{(1-\epsilon)^2} p_s^2 \ge \tilde p_s^2 \ge (1-\epsilon)^2 p_s^2$$

and hence

$$-\epsilon p_s \le \tilde p_s - p_s \le \frac{\epsilon}{1-\epsilon} p_s.$$

□

Now, we are in position to prove Theorem 1.2.

Proof. As it follows from Lemma 3.1

$$T^{(s)}(c) = T(c) - q_s - p_s^2 \int \frac{d\nu^{(s)}(x)}{T(c) - x}.$$

Here $\nu^{(s)}$ is a discrete measure such that $\operatorname{supp}\{\nu^{(s)}\} \subset [-\xi, \xi]$, $\nu^{(s)}([-\xi,\xi]) = 1$, recall that $[-\xi,\xi]$ is the smallest interval containing the Julia set E. In particular,

$$T^{(sd_n)}(c) = T(c) - q_0 - p_{sd_n}^2 \int \frac{d\nu^{(sd_n)}(x)}{T(c) - x}.$$

Now, since

$$\operatorname{dist}_{\{c: T'(c) = 0\}}\{T(c), [-\xi,\xi]\} = \delta > 0,$$

for every $\epsilon > 0$ there exists n such that

$$(1-\epsilon) \le \frac{T^{(sd_n)}(c)}{T(c) - q_0} \le (1-\epsilon)^{-1}$$

(here we used Lemma 3.2). Recall (25), so, Lemma 3.4 completes the proof.

□

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